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Topic 6:
Binomial Expansions (4047)

THE ABOUT

CHAPTER ANALYSIS

MASTERY

- Relatively straight forward chapter, but there have been variants of this chapter that prove to be quite challenging for students that are not conceptually strong
- 1 **key** concept

EXAM

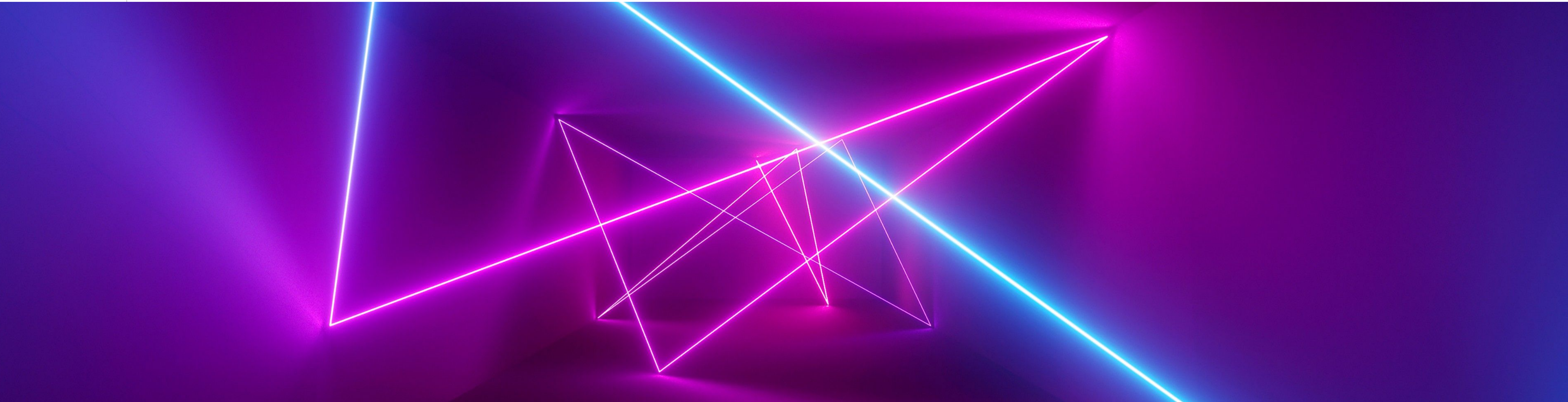
- Concepts usually tested as a stand-alone topic

WEIGHTAGE

- High medium weightage
- Tested consistently every year
- Typically, an 7m question, 1 question in one of the papers

KEY CONCEPT

Binomial Theorem



Binomial Theorem

An expression which contains two terms

Can be written in 3 different ways:

Expression 1:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{r} a^{n-r} b^r$$

Expression 2:

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_r a^{n-r} b^r$$

Expression 3:

$$(a+b)^n = a^n + (n) a^{n-1} b + \left(\frac{n(n-1)}{2}\right) a^{n-2} b^2 + \cdots + \left(\frac{n(n-1)\ldots(n-r+1)}{(1)(2)\ldots(r)}\right) a^{n-r} b^r$$

Important

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

where $n! = n(n-1)(n-2)\ldots(2)(1)$

This is a rather important result as it can be tested with questions where the binomial coefficients are not given

Expression	Special Terminology
${}^nC_0, {}^nC_1, {}^nC_2, \ldots$	Binomial Coefficients
${}^nC_r a^{n-r} b^r$	General $(r+1)^{\text{th}}$ term
$n!$	n factorial
x^0	Independent term of x

Example:

Find the coefficient of x^9 of the expansion of

$$\left(2x^2 - \frac{1}{x}\right)^{12}$$

Solution

We first find the $(r+1)^{\text{th}}$ term

$$(r+1)^{th} \text{ term} = \binom{12}{r}(2x^2)^{12-r}\left(-\frac{1}{x}\right)^r$$

$$= \binom{12}{r}(2)^{12-r}(-1)^r(x^2)^{12-r}(x)^{-r}$$

$$= \binom{12}{r}(2)^{12-r}(-1)^r(x)^{24-3r}$$

For the term in x^9, by comparison,

$24 - 3r = 9$

$r = 5$

At this junction, if you get a r value where it is NOT an integer, please review your workings again as this is not possible

$\therefore$ The coefficient of $x^9 = \binom{12}{5}(2)^{12-5}(-1)^5$

$= -101376$

Take Note:

When the question is asking for "Independent term of x" or "Constant term", follow the same steps as shown but when comparing the x powers, equate the expression to $\mathbf{0}$

$$24 - 3r = 0$$

Take Note:

Observe the way the groupings have been done. We group the constants (i.e. numbers) and variables (i.e. x) separately

For the term

$$(2x^2)^{12-r} = (2)^{12-r}(x^2)^{12-r}$$

We split up the constants and variables using Indices Law from **Chapter 2: Indices & Surds**

For the term

$$\left(-\frac{1}{x}\right)^r = (-1)^r(x)^{-r}$$

We split up the constants and variables as such by first moving the variables to the numerator via the negative power trick (Indices Law from **Chapter 2: Indices & Surds**)

Always factor out the (-1) if there is a negative sign in the expression

Example: [Common Exam Question Type]

Find, in ascending powers of x, the first three terms of the expansion of

$$\left(\frac{1}{3x}-2x^2\right)^6$$

***Hence, or otherwise**, find the term independent of x in the expansion of

$$\left(\frac{1}{3x}-2x^2\right)^6(5+x^3)$$

Solution

$$\left(\frac{1}{3x}-2x^2\right)^6=\left(\frac{1}{3x}\right)^6+\binom{6}{1}\left(\frac{1}{3x}\right)^5(-2x^2)+\binom{6}{2}\left(\frac{1}{3x}\right)^4(-2x^2)^2+\cdots$$

$$=\frac{1}{729}x^{-6}-\frac{4}{81}x^{-3}+\frac{20}{27}+\cdots$$

The independent term of x for the expansion will be

$$\left(\frac{1}{3x}-2x^2\right)^6(5+x^3)=\left(\frac{1}{729}x^{-6}-\frac{4}{81}x^{-3}+\frac{20}{27}+\cdots\right)(5+x^3)$$

$$=\cdots+\left(-\frac{4}{81}x^{-3}\right)(x^3)+\left(\frac{20}{27}\right)(5)+\cdots$$

$$=3\frac{53}{81}$$

Explanation & Methodology:

The second part of this question is commonly tested in exams where students will be required to find another term of x in another similar expansion

$$\left(\frac{1}{3x}-2x^2\right)^6\underbrace{(5+x^3)}_{\text{Additional Term}}$$

Always use the additional term added to help you solve the question. Many students will disregard the implications of that additional term

Since we are required to find the independent term of x, students should ask themselves which term multiplied by each of the terms in the additional term is required to get the independent term of x. Each of the unique terms in the additional term **MUST** be used once.

$$\left(\frac{1}{729}x^{-6}-\frac{4}{81}x^{-3}+\frac{20}{27}+\cdots\right)(5+x^3)$$

To use the x^3 term, we need to use the x^{-3}

To use the constant 5, we need to use the other constant

Example: [Challenging Variant]

In the given expansion, where $n > 0$, the first 3 terms in ascending powers of x are

$$\left(\frac{k}{x^3} + cx\right)^n = \frac{1}{x^{66}} + \frac{44}{x^{62}} + \frac{924}{x^{58}} + \cdots$$

Find the values of k, n and c

[S4 CCHS(Y) P2/2012 PRELIM Qn 2(a)]

Solution

To find the values, we expand the left-hand side of the equation first

$$\left(\frac{k}{x^3} + cx\right)^n = \left(\frac{k}{x^3}\right)^n + \binom{n}{1}\left(\frac{k}{x^3}\right)^{n-1}(cx)^1 + \binom{n}{2}\left(\frac{k}{x^3}\right)^{n-2}(cx)^2 + \cdots$$

$$= \frac{k^n}{x^{3n}} + n\left(\frac{k^{n-1}}{x^{3n-3}}\right)(cx) + \frac{n(n-1)}{2}\left(\frac{k^{n-2}}{x^{3n-6}}\right)(c^2x^2) + \cdots$$

$$= \frac{k^n}{x^{3n}} + \frac{nck^{n-1}}{x^{3n-4}} + \frac{n(n-1)c^2k^{n-2}}{2x^{3n-6}} + \cdots$$

Continued

Comparing coefficients,

$x^{3n} = x^{66}$
$3n = 66$
$n = 22$

$k^{22} = 1$
$k = 1$

$nck^{n-1} = 44$
$(22)(c)(1) = 44$
$c = 2$

Explanation:

This variant is challenging as the left-hand side of the binomial expression has unknowns. When dealing with such unknowns, students must be able to apply the Binomial Theorem, and also use the given result below to solve for all the Binomial Coefficients

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$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