A LEVEL
H2 MATHEMATICS
VECTORS

CHAPTER ANALYSIS

MASTERY

- 3D Vectors, Vector Algebra
- Scalar (Dot) and Vector (Cross) Product
- Vector Equations for Lines
- Vector Equations for Planes

EXAM

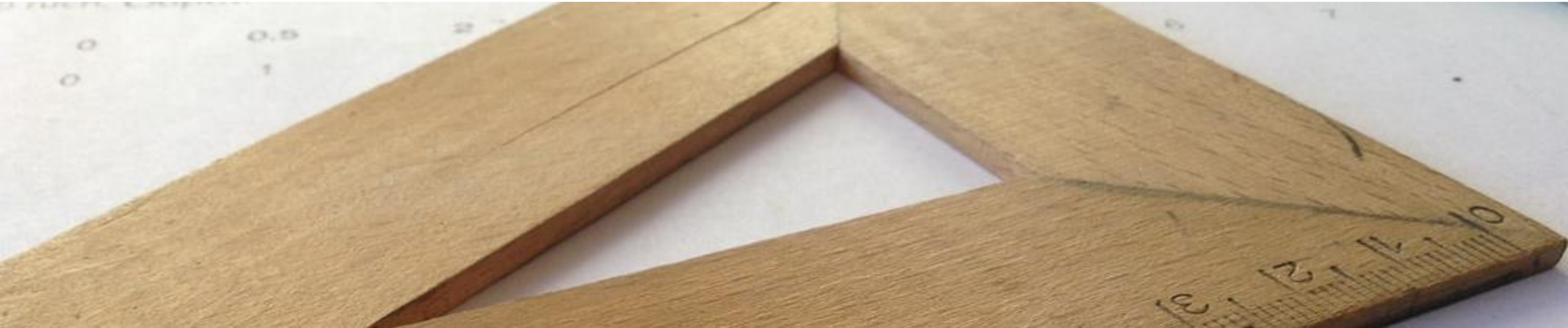
- Important to understand concepts instead of blindly memorizing
- Good to draw out diagrams to aid understanding
- Unfortunately, practice makes perfect. Make sure to practice the hard questions.

WEIGHTAGE

- Huge topic, tested every year without fail
- Minimally 2 questions a year on Vectors
- Typically constitutes about 10% of final grade, much higher weightage as compared to other chapters

VECTORS I

BASIC VECTOR PROPERTIES
VECTOR ALGEBRA
PARALLEL AND NON-PARALLEL VECTORS
SCALAR PRODUCT (DOT PRODUCT)
VECTOR PRODUCT (CROSS PRODUCT)



Basic Vector Properties

$$|\overrightarrow{OA}| = |\boldsymbol{a}| = \sqrt{4^2 + 7^2 + 11^2}$$

$$\overrightarrow{OA} = \boldsymbol{a} = \begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix} = 4\boldsymbol{i} + 7\boldsymbol{j} + 11\boldsymbol{k}$$

The **zero/null vector** is a vector with zero magnitude and no direction.

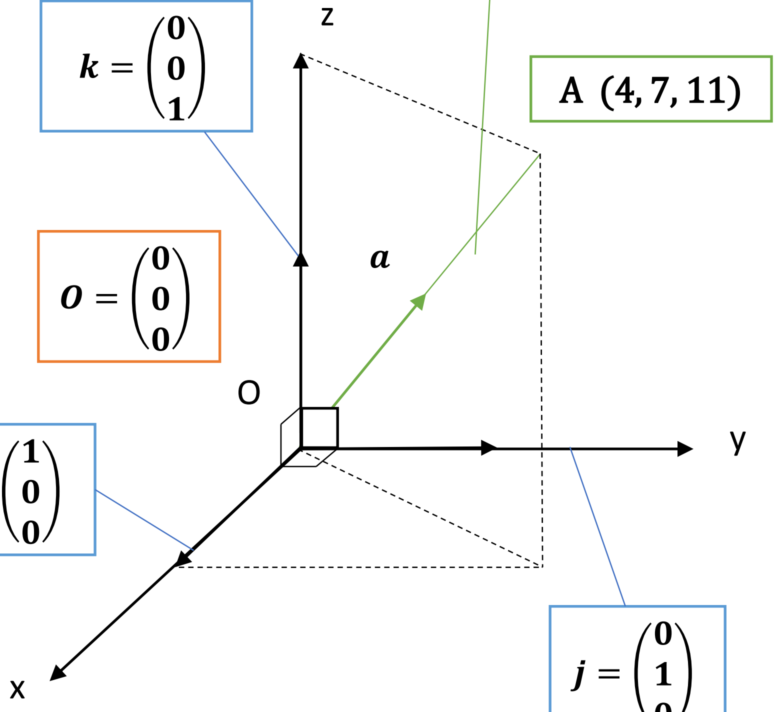
$$\boldsymbol{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A $(4, 7, 11)$

$\boldsymbol{O}$ is the **origin**, from which we usually associate our position vectors.

$$\boldsymbol{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

A **unit vector,** denoted by $\hat{\boldsymbol{a}}$, is a vector whose **magnitude is 1**.

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

$\boldsymbol{i}, \boldsymbol{j}$ ***and*** $\boldsymbol{k}$ are **unit vectors.**

- A scalar quantity has magnitude but no associated direction (e.g. distance and speed).
- A **vector** quantity has both **magnitude** and **direction** (e.g. displacement and velocity).
- A *position vector* defines the position of a point relative to another.
- A *free/displacement vector* is a vector with no associated position.

Modulus of Vector = Magnitude or Length of Vector

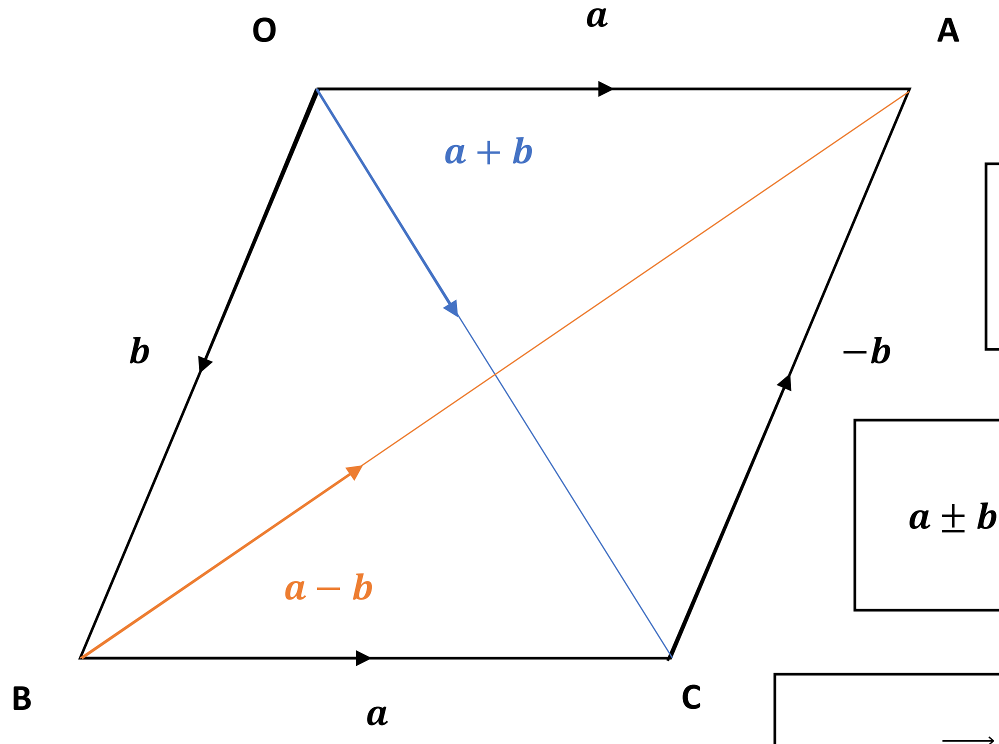
If $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then

$$|a| = \sqrt{x^2 + y^2 + z^2}$$

Vector Algebra

$$a + b = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OB} + \overrightarrow{BC}$$

In general, $\overrightarrow{UV} = \overrightarrow{OV} - \overrightarrow{OU}$



$$a - b = a + (-b) = \overrightarrow{BO} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$

The **negative of a vector** has the same magnitude as a vector but is opposite in direction (i.e. ***a*** and ***−a***).

$$a \pm b = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \pm \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \\ z_1 \pm z_2 \end{pmatrix}$$

If $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$

$$|\overrightarrow{AB}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Equal Vectors

Vectors are **equal** when they have the same direction and magnitude. If $\overrightarrow{AB} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, $\overrightarrow{AB} = \overrightarrow{CD}$, then $\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ and $d = p$, $e = q$, and $f = r$

Scalar Multiplication

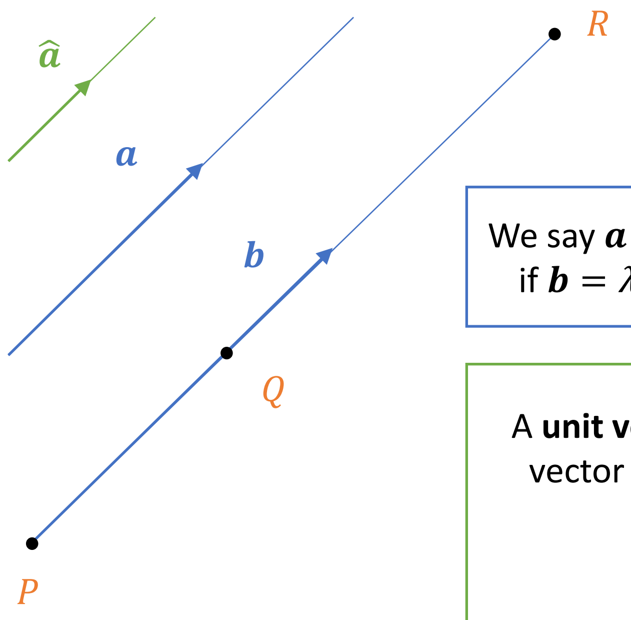
When vector $\boldsymbol{a}$ is multiplied by the scalar λ, the magnitude of the vector changes and the vector $\lambda\boldsymbol{a}$ has magnitude λ times of $\boldsymbol{a}$ (i.e. $|\lambda\boldsymbol{a}| = \lambda|\boldsymbol{a}|$)

- If $\lambda > 0$, $\lambda\boldsymbol{a}$ and $\boldsymbol{a}$ are in the same direction
- If $\lambda = 0$, $\lambda\boldsymbol{a}$ is a zero vector i.e. $\lambda\boldsymbol{a} = \boldsymbol{0}$
- If $\lambda < 0$, $\lambda\boldsymbol{a}$ and $\boldsymbol{a}$ are in opposite directions

Laws of Vector Algebra

1. $a + b = b + a$
2. $\lambda a = a\lambda$
3. $(a + b) + c = a + (b + c)$
4. $(\lambda\mu)a = \lambda(\mu a)$
5. $(\lambda + \mu)a = \lambda a + \mu a$
6. $\lambda(a + b) = \lambda a + \lambda b$

Parallel & Non-Parallel Vectors



We say $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ if and only if $\boldsymbol{b} = \lambda\boldsymbol{a}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$

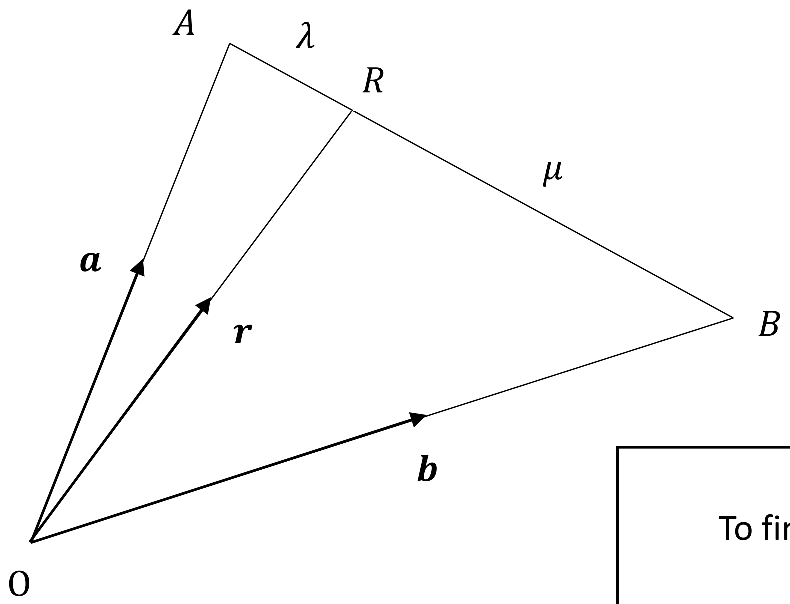
A **unit vector,** denoted by $\hat{\boldsymbol{a}}$, is a vector whose **magnitude is 1**.

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Collinearity

Three points, P, Q and R are collinear if and only if $\overrightarrow{\boldsymbol{PQ}} \,//\, \overrightarrow{\boldsymbol{PR}}$, with P as the common point i.e. $\overrightarrow{\boldsymbol{PQ}} = \lambda\overrightarrow{\boldsymbol{PR}}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$

The Ratio Theorem



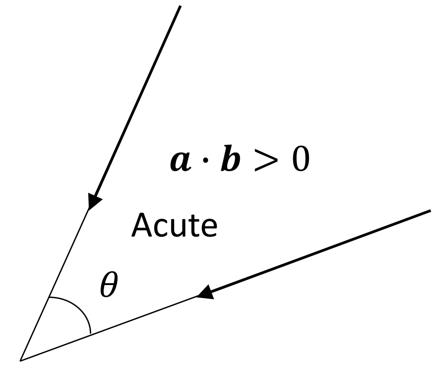
If R is the **midpoint** of AB, then R divides AB in the ratio **1:1** and

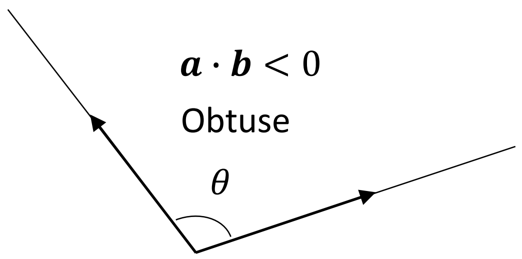
$$\overrightarrow{\boldsymbol{OR}} = \boldsymbol{r} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

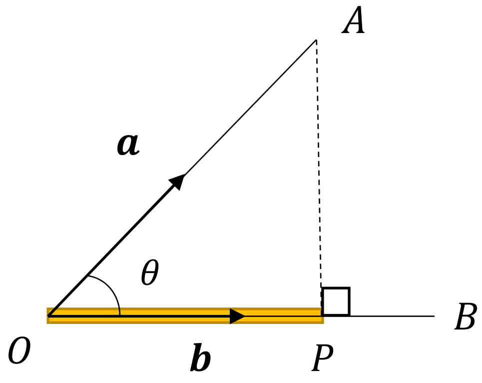
To find $\overrightarrow{\boldsymbol{OR}} = \boldsymbol{r}$:

$$\boldsymbol{r} = \frac{\mu\boldsymbol{a} + \lambda\boldsymbol{b}}{\lambda + \mu}$$

Scalar Product (Dot Product)







The scalar product of two vectors **a** and **b**, is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are either **both leaving from** or **both meeting at** the same point

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$\boldsymbol{a} \cdot \boldsymbol{b}$ is called a scalar product because the product is a scalar

Perpendicular Vectors

Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** i.e. $\boldsymbol{a} \perp \boldsymbol{b}$, if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$

Scalar Product Properties

1. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
2. $\boldsymbol{a} \cdot (\boldsymbol{b} \pm \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b}) \pm (\boldsymbol{a} \cdot \boldsymbol{c})$
3. $\lambda\,(\boldsymbol{a} \cdot \boldsymbol{b}) = (\lambda \boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (\lambda \boldsymbol{b})$
4. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Length of Projection

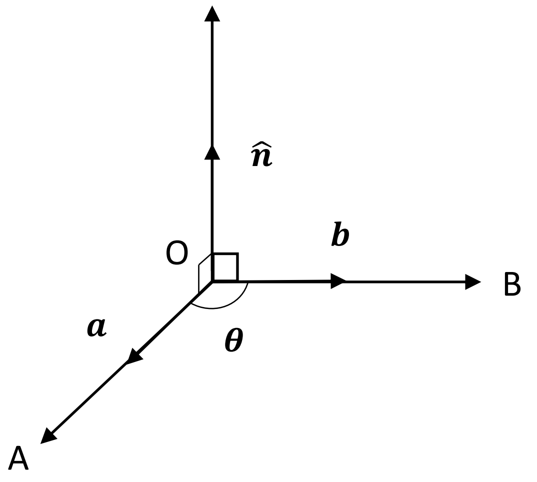
$$|\overrightarrow{OP}| = |\boldsymbol{a} \cdot \hat{\boldsymbol{b}}| = \frac{|\boldsymbol{a} \cdot \boldsymbol{b}|}{|\boldsymbol{b}|}$$

Angle Between Two Non-Zero Vectors

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

→ Link this to **direction cosines**, which is the cosine of the angle between a vector and the x- , y- and z-axes

Vector Product (Cross Product)



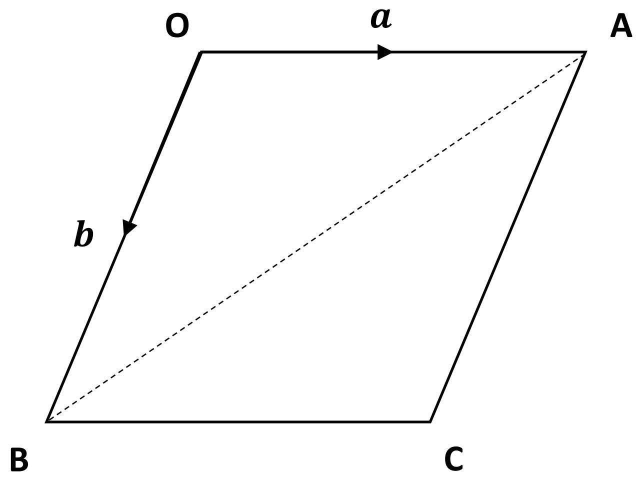
The vector product of two vectors **a** and **b**, is defined as

$$\boldsymbol{a} \times \boldsymbol{b} = (|\boldsymbol{a}||\boldsymbol{b}| \sin\theta)\hat{\boldsymbol{n}}$$

θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\hat{\boldsymbol{n}}$ is the unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ (unit vector of normal)

$\boldsymbol{a} \times \boldsymbol{b}$ is called a vector product because the product is a vector

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$$



Parallel Vectors

Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **parallel** if and only if $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$

Area of Triangle OAB = $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$

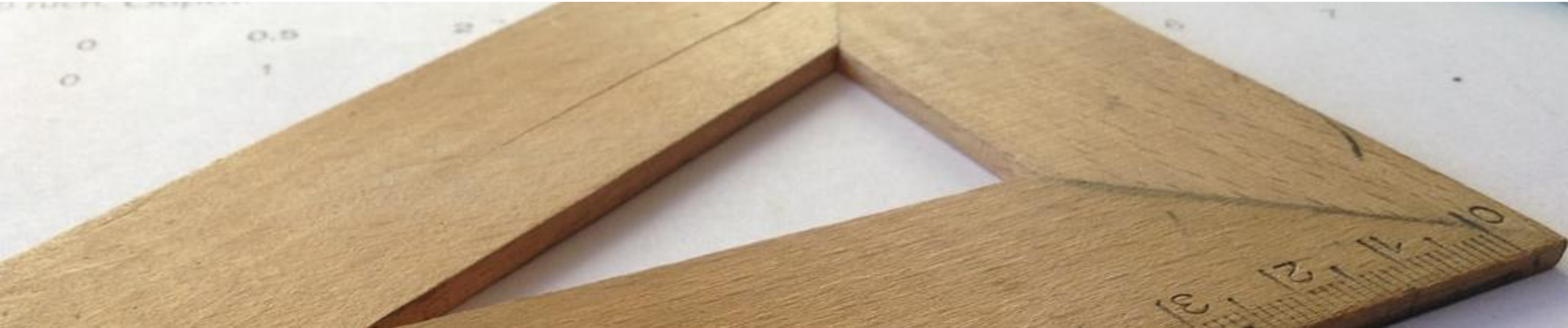
Area of Parallelogram OACB = $|\boldsymbol{a} \times \boldsymbol{b}|$

Vector Product Properties

1. $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$
2. $\boldsymbol{a} \times (\boldsymbol{b} \pm \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \pm (\boldsymbol{a} \times \boldsymbol{c})$
3. $\boldsymbol{\lambda}(\boldsymbol{a} \times \boldsymbol{b}) = (\lambda \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\boldsymbol{\lambda b})$
4. $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$
5. $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{b} \times \boldsymbol{a}| = |\boldsymbol{a}||\boldsymbol{b}||\sin\theta|$

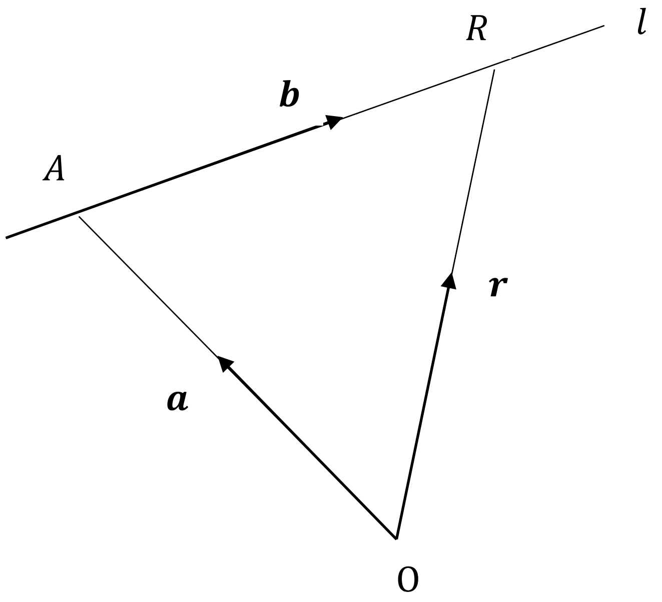
VECTORS II

EQUATIONS OF STRAIGHT LINES
CALCULATIONS FOR A POINT AND A LINE
CALCULATIONS FOR A PAIR OF LINES



Equations of Straight Lines

Vector Equation



$$l : \boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}, \qquad \lambda \in \mathbb{R}$$

$\boldsymbol{a}$: Any point on the line

$\boldsymbol{b}$: Direction vector of line

Parametric Equation

$$\begin{cases} x = \boldsymbol{a_1} + \lambda \boldsymbol{b_1} \\ y = \boldsymbol{a_2} + \lambda \boldsymbol{b_2} \\ z = \boldsymbol{a_3} + \lambda \boldsymbol{b_3} \end{cases}, \qquad \lambda \in \mathbb{R}$$

From vector equation, let $\mathrm{r} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}$ to transform into parametric equation

Make λ the subject

Cartesian Equation

$$\begin{cases} \lambda = \dfrac{\boldsymbol{x} - \boldsymbol{a_1}}{\boldsymbol{b_1}} \\ \lambda = \dfrac{\boldsymbol{y} - \boldsymbol{a_2}}{\boldsymbol{b_2}} \\ \lambda = \dfrac{\boldsymbol{z} - \boldsymbol{a_3}}{\boldsymbol{b_3}} \end{cases}, \qquad \lambda \in \mathbb{R}$$

Equate λ

$$\frac{\boldsymbol{x} - \boldsymbol{a_1}}{\boldsymbol{b_1}} = \frac{\boldsymbol{y} - \boldsymbol{a_2}}{\boldsymbol{b_2}} = \frac{\boldsymbol{z} - \boldsymbol{a_3}}{\boldsymbol{b_3}}, \qquad \lambda \in \mathbb{R}$$

If $\boldsymbol{b_1} = \boldsymbol{0}$, cartesian equations becomes

$$x = \boldsymbol{a_1}, \frac{\boldsymbol{y} - \boldsymbol{a_2}}{\boldsymbol{b_2}} = \frac{\boldsymbol{z} - \boldsymbol{a_3}}{\boldsymbol{b_3}}$$

Calculations For A Point And A Line

Determine If A Point Lies On A Line

To determine if a point lies on a line, substitute the point into the vector equation of the line and solve for a unique value of λ

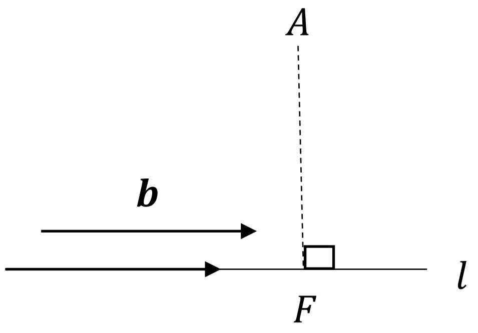
If a unique solution is found, the point lies on the line. If not, it does not lie on the line.

Point A (2, -3, 9) lies on the line $l_1 : \boldsymbol{r} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-6} \\ \mathbf{3} \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{6} \end{pmatrix}$ because $\begin{pmatrix} \mathbf{2} \\ \mathbf{-3} \\ \mathbf{9} \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-6} \\ \mathbf{3} \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{6} \end{pmatrix}$ satisfies the equation l_1 with $\lambda = 1$ (unique solution).

Perpendicular Distance From A Point To A Line

1. Find foot of the perpendicular from point to line (see right side)
2. Find modulus of $\overrightarrow{\boldsymbol{OF}}$

Foot Of Perpendicular From A Point To A Line



To find foot of perpendicular $\overrightarrow{\boldsymbol{OF}}$ from Point A to line l :

1. Let F be foot of perpendicular from point to line
2. Since F lies on l , let $\overrightarrow{\boldsymbol{OF}}$ = vector equation of line
3. Since $\overrightarrow{\boldsymbol{AF}} \perp l$, let $\overrightarrow{\boldsymbol{AF}} \cdot \boldsymbol{b} = \mathbf{0}$ to find value of λ
4. Sub back value of λ into l to find $\overrightarrow{\boldsymbol{OF}}$

To find foot of perpendicular $\overrightarrow{\boldsymbol{OF}}$ from Point A (2, -3, 9) to line $l_1 : \boldsymbol{r} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-6} \\ \mathbf{3} \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{6} \end{pmatrix}$:

1. Let F be foot of perpendicular from point to line
2. Since F lies on l , let $\overrightarrow{\boldsymbol{OF}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-6} \\ \mathbf{3} \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{6} \end{pmatrix}$
3. Since $\overrightarrow{\boldsymbol{AF}} \perp l$, let $\overrightarrow{\boldsymbol{AF}} \cdot \boldsymbol{b} = \left[\begin{pmatrix} \mathbf{1+\lambda} \\ \mathbf{-6+3\lambda} \\ \mathbf{3+6\lambda} \end{pmatrix} - \begin{pmatrix} \mathbf{2} \\ \mathbf{-3} \\ \mathbf{9} \end{pmatrix} \right] \cdot \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{6} \end{pmatrix} = \mathbf{0} \Rightarrow \lambda = -\frac{23}{13}$
4. Sub λ into l to find $\overrightarrow{\boldsymbol{OF}} \Rightarrow \overrightarrow{\boldsymbol{OF}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-6} \\ \mathbf{3} \end{pmatrix} - \frac{23}{13} \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{6} \end{pmatrix} = -\frac{1}{13} \begin{pmatrix} \mathbf{10} \\ \mathbf{147} \\ \mathbf{99} \end{pmatrix}$

Calculations For A Pair Of Lines

Relationship Between Two Lines

	Parallel	Not Parallel
Intersecting	Same Line	Intersecting
Not Intersecting	Parallel Lines	Skew Lines

Check if lines are **parallel** to one another using direction vectors → If vectors are **not parallel**, check if they **intersect** by treating their vector equations as **simultaneous equations** → If **no solution** is found, lines are **not parallel**, **do not intersect** and are **skew**

Parallel

$$l_1 : \boldsymbol{r} = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \qquad l_2 : \boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \boldsymbol{\mu} \begin{pmatrix} 2 \\ 6 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 6 \\ 12 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \Rightarrow l_1 \, // \, l_2$$

Intersecting

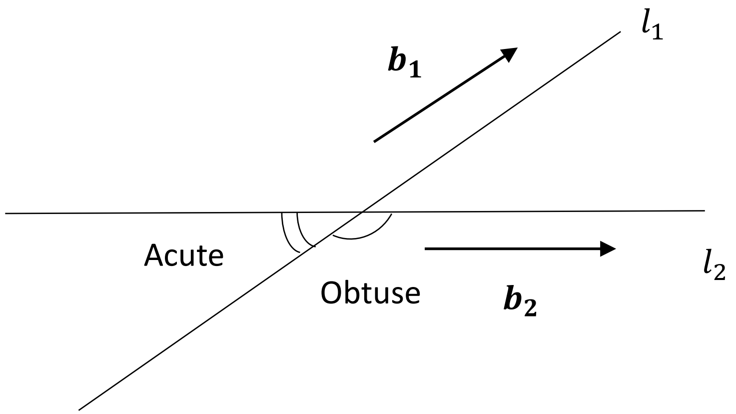
$$\begin{pmatrix} 4 \\ 8 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\begin{cases} 4 + \lambda = 7 + 6\mu \\ 8 + 2\lambda = 6 + 4\mu \\ 3 + \lambda = 5 + 5\mu \end{cases}$$

Using G.C. to solve $\Rightarrow \lambda = -3$ and $\mu = -1$

Angle Between Two Non-Zero Vectors

$$\cos\theta = \frac{\boldsymbol{b_1} \cdot \boldsymbol{b_2}}{|\boldsymbol{b_1}||\boldsymbol{b_2}|}$$



Angle Between Two Non-Zero Vectors

θ may be acute or obtuse,
depending on the sign of the direction vectors
If θ is acute, then the acute angle is θ
If θ is obtuse, then the acute angle is 180° - θ

VECTORS II

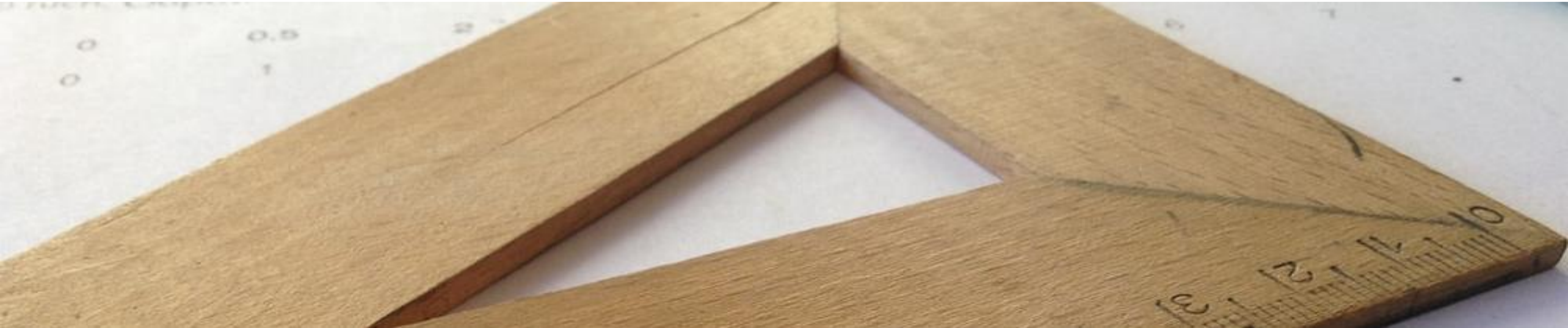
EQUATIONS OF PLANES

CALCULATIONS FOR A LINE AND A PLANE

CALCULATIONS FOR A POINT AND A PLANE

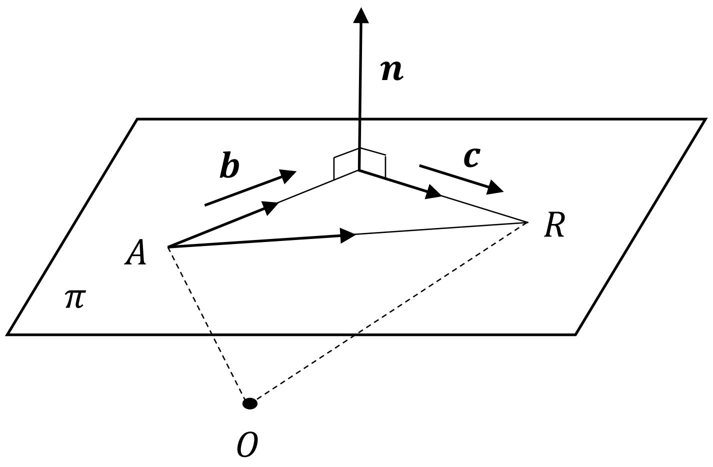
CALCULATIONS FOR TWO PLANES

CALCULATIONS FOR THREE PLANES



Equations of Planes

Vector Equation



You may use the vector equation of a plane to find any point on the plane i.e. $\mathbf{r} = \overrightarrow{\boldsymbol{OR}}$

$$\pi : \boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}, \qquad \lambda, \mu \in \mathbb{R}$$

- $\boldsymbol{a}$: Any point on plane
- $\boldsymbol{b}$, $\boldsymbol{c}$: Direction vector of lines on plane

Scalar Product Form

$$\pi : \boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n} = d \text{ (scalar)}$$

- $\boldsymbol{n}$: Normal vector of plane
- $\boldsymbol{a}$: Any point on plane

To find normal vector $\boldsymbol{n}$:
Cross product any two direction vectors on the plane i.e. $\boldsymbol{b} \times \boldsymbol{c}$

Cartesian Equation

Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

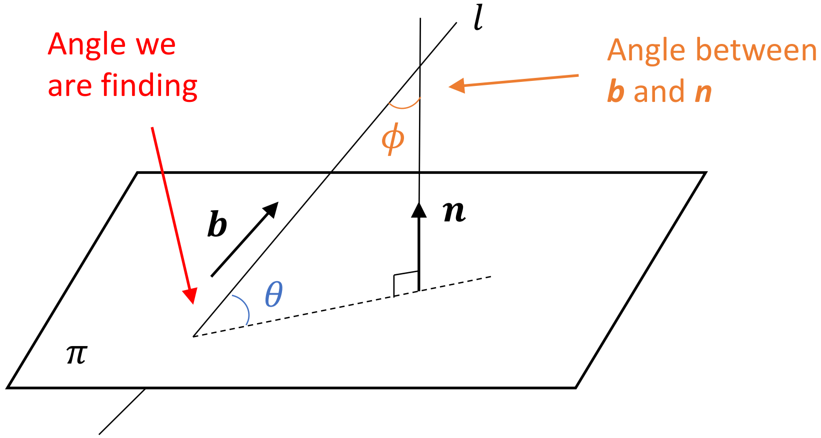
$$\pi : \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{n_1} \\ \boldsymbol{n_2} \\ \boldsymbol{n_3} \end{pmatrix} = d$$

$$\pi : \boldsymbol{x}\boldsymbol{n_1} + \boldsymbol{y}\boldsymbol{n_2} + \boldsymbol{z}\boldsymbol{n_3} = d$$

Where $\mathrm{d} = a_1 \boldsymbol{n_1} + a_2 \boldsymbol{n_2} + a_3 \boldsymbol{n_3}$

Calculations For A Line And A Plane

Angle Between A Line And A Plane

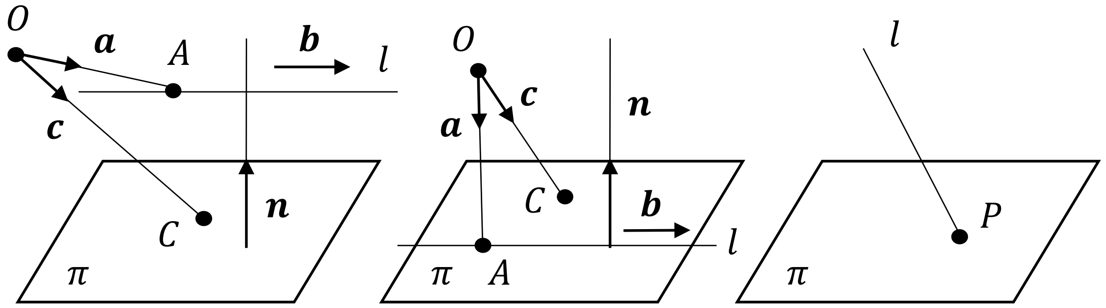


Let θ is the acute angle between the line l and the plane π. To find θ we can first find ϕ, the acute angle between l and the normal $\boldsymbol{n}$ to the plane π, which is also the angle between $\boldsymbol{b}$ and $\boldsymbol{n}$. Then $\theta = 90° - \phi$

1. Use $\cos \phi = \frac{\boldsymbol{b} \cdot \boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}$ to find ϕ
2. Then $\theta = 90° - \phi$
3. If $\theta > 90° \Rightarrow \theta - 90°$ to find acute angle between l and π

Point Of Intersection Between A Line And A Plane

There are 3 possible scenarios for the relationship between l and π:



1. l and π do not intersect
2. l lies within π
3. l and π intersect at a point

To determine the relationship between l and π:

Check if the line is parallel to the plane i.e. if $\boldsymbol{n}$ is perpendicular to $\boldsymbol{b}$ $\Rightarrow \boldsymbol{b} \cdot \boldsymbol{n} = 0$

- **Yes** → Check if $\boldsymbol{a} \cdot \boldsymbol{n} = \boldsymbol{c} \cdot \boldsymbol{n}$
  - **Yes** → **Scenario 2**
  - **No** → **Scenario 1**
- **No** → **Scenario 3**
  Find point of intersection by solving simultaneous equations of l and π

Calculations For A Point And A Plane

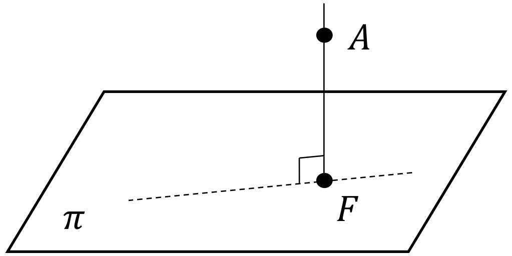
Determine If A Point Lies On A Plane

To determine if a point lies on a line, substitute the point into the scalar product equation of the plane and observe if $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{d}$

Perpendicular Distance From A Point To A Plane

1. Find foot of the perpendicular from point to plane (see right side)
2. Find modulus of $\overrightarrow{\boldsymbol{OF}}$

Foot Of Perpendicular From A Point To A Plane



To find foot of perpendicular $\overrightarrow{\boldsymbol{OF}}$ from Point A to plane π:

1. Let F be foot of perpendicular from point to plane
2. Let l_{AF} be a line with $\boldsymbol{r} = \overrightarrow{\boldsymbol{OA}} + \lambda\boldsymbol{n}$ where $\boldsymbol{n}$ is the normal vector of the plane
3. Since l_{AF} intersects the plane, let $l_{AF} \cdot \boldsymbol{n} = \boldsymbol{d}$ to find value of λ
4. Sub value of λ into l_{AF} to find $\overrightarrow{\boldsymbol{OF}}$

To find foot of perpendicular $\overrightarrow{\boldsymbol{OF}}$ from Point A (2, -3, 9) to plane π : $\boldsymbol{r} \cdot \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-3} \end{pmatrix} = 1$:

1. Let F be foot of perpendicular from point to plane
2. Let $l_{AF} : \boldsymbol{r} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-3} \\ \mathbf{9} \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-3} \end{pmatrix}$
3. Since l_{AF} intersects the plane, let $l_{AF} \cdot \boldsymbol{n} = 1$
   $\left[\begin{pmatrix} \mathbf{2} + \lambda \\ \mathbf{-3} + \lambda \\ \mathbf{9} - 3\lambda \end{pmatrix}\right] \cdot \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-3} \end{pmatrix} = 1 \Rightarrow \lambda = \frac{29}{11}$
4. Sub λ into l_{AF} to find $\overrightarrow{\boldsymbol{OF}} \Rightarrow \overrightarrow{\boldsymbol{OF}} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-3} \\ \mathbf{9} \end{pmatrix} + \frac{29}{11} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-3} \end{pmatrix} = \frac{1}{11} \begin{pmatrix} \mathbf{51} \\ \mathbf{-4} \\ \mathbf{12} \end{pmatrix}$

Calculations For Two Planes

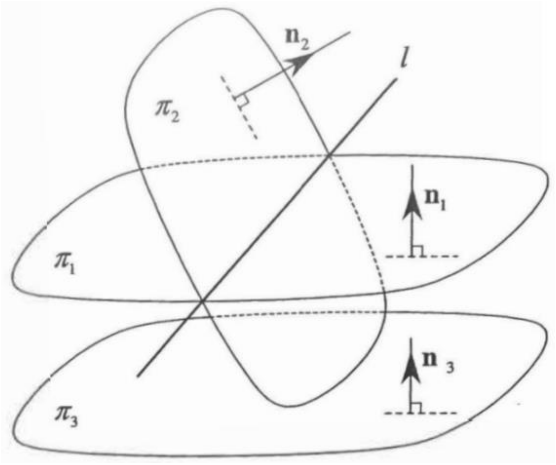
Intersection Of Two Planes

Case 1: Do Not Intersect, Parallel Planes

$\boldsymbol{n_1} = \lambda\, \boldsymbol{n_3}$ for some $\lambda \in \mathbb{R}$

Case 2: Two Planes Intersect In A Line

Use G.C. to solve equations for 2 planes using simultaneous equations to get 2 equations containing z. Let z = t and manipulate equations to get equation of a line.



Case 2 Example

$$\pi_1: x + 3y + 2z = 4$$
$$\pi_2: x - y - z = 4$$

Using G.C. and letting z = t:

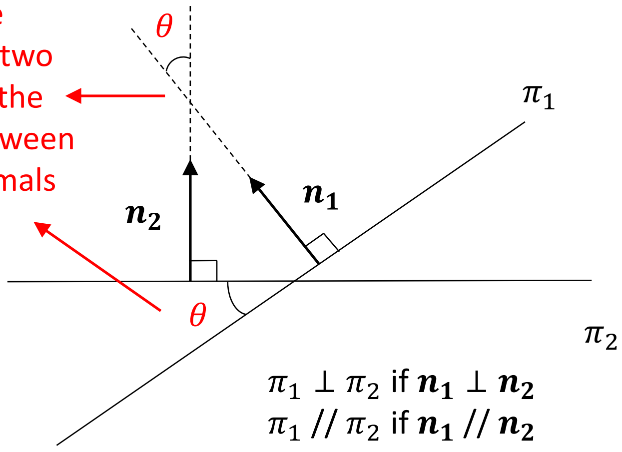
$$x = 4 + \frac{1}{4}t$$
$$y = -\frac{3}{4}t$$
$$z = t$$

$$\Rightarrow \quad \boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ 1 \end{pmatrix}, t \in \mathbb{R}$$

Angle Between Two Planes

$$\cos\theta = \frac{\boldsymbol{n_1} \cdot \boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}$$

The angle between two planes is the angle between their normals



$\pi_1 \perp \pi_2$ if $\boldsymbol{n_1} \perp \boldsymbol{n_2}$

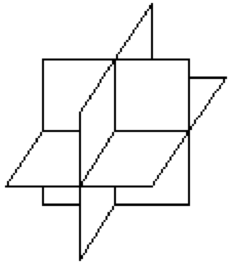
$\pi_1 \,//\, \pi_2$ if $\boldsymbol{n_1} \,//\, \boldsymbol{n_2}$

Angle Between Two Planes

θ may be acute or obtuse,
depending on the sign of the direction vectors
If θ is acute, then the acute angle is θ
If θ is obtuse, then the acute angle is 180° - θ

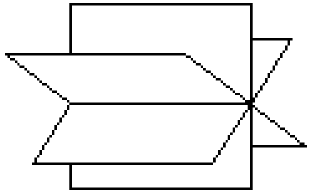
Calculations For Three Planes

Relationship Between Three Planes



Case 1: Intersect At A Single Point

Use G.C. to solve equations for 3 planes using simultaneous equations to get unique values for x, y and z



Case 2: Intersect Along One Common Line

Use G.C. to solve equations for 3 planes using simultaneous equations to get 3 equations containing z. Manipulate equations to get equation of a line.

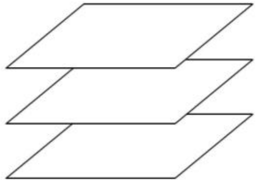
Case 2 Example

$$\left.\begin{aligned}\pi_1&: 2x + 3y - 3z = 14\\ \pi_2&: -3x + y + 10z = -32\\ \pi_3&: x + 7y + 4z = -4\end{aligned}\right\}$$

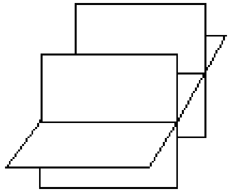
Using G.C. and letting z = t:

$$\begin{aligned}x &= 10 + 3t\\ y &= -2 - t\\ z &= t\end{aligned} \quad \Rightarrow \quad \boldsymbol{r} = \begin{pmatrix}\mathbf{10}\\ \mathbf{-2}\\ \mathbf{0}\end{pmatrix} + t\begin{pmatrix}\mathbf{3}\\ \mathbf{-1}\\ \mathbf{1}\end{pmatrix}, t \in \mathbb{R}$$

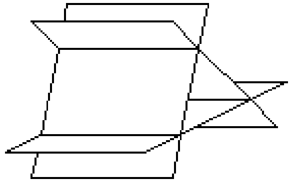
Case 3: Do Not Intersect At Any Common Point Or Line



(a) All 3 Parallel



(b) 2 Of The 3 Parallel



(c) Triangular Prism

Use G.C. to solve equations for 3 planes using simultaneous equations, obtains no solution found.

Check if planes are **parallel** to one another using normal vectors (Case 3a or 3b)

→ If planes are **not parallel**, check if they **intersect** by treating equations as **simultaneous equations** (Case 1 or 2)

→ If **no solution** is found, lines are **not parallel**, **do not intersect** and planes form a triangular prism (Case 3c)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