A LEVEL
H2 MATHEMATICS
FUNCTIONS

CHAPTER ANALYSIS

MASTERY

- Identify functions, and their domain and range
- Existence of inverse and composite functions
- Finding inverse and composite functions, their domains and ranges
- Relationship between a function and its inverse
- Domain restrictions to obtain inverse and composite functions

EXAM

- Heavy emphasis on inverse functions, followed by composite functions
- Commonly tested with graphing techniques

WEIGHTAGE

- Appears every year, at least 1 question
- Typically constitutes 3-5% of final grade

Relations & Functions

A **relation** is a rule that maps elements of a given set A (input) to elements of a given set B (output). Set A is known as the domain and set B the codomain of the relation.

A **function** f : X |→ Y is a relation which maps each element *x* in the set X to **one and only one** element *y* in the codomain Y (i.e. every input has exactly one output). Functions are subsets of relations.

Domain (X-Values)

Set X, the set of x values is called the **domain** of f, denoted by D_f. This is usually defined in the question.

Range (Y-Values)

If the element *a* ∈ X is mapped to the element *b* ∈ Y, then we write f (*a*) = *b*. *b* is called the image of *a* under f.

The **range** of f is the set of images of X under f, denoted by R_f (i.e. the set of y values found when the domain is put into the function).

Representing Functions

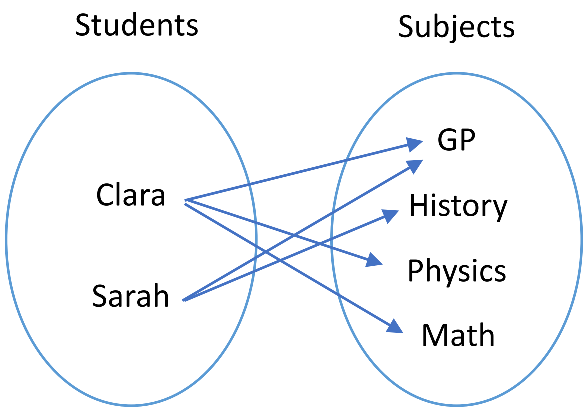
To define a function, both the rule and domain have to be clearly stated. It is not necessary to state the range of the function. At 'A' Levels most functions considered are relations between real numbers.

E.g. we write:

(1) f : x |→ x + 2, x ∈ ℝ+

(2) f (x) = x + 2, x ∈ ℝ+

Relations

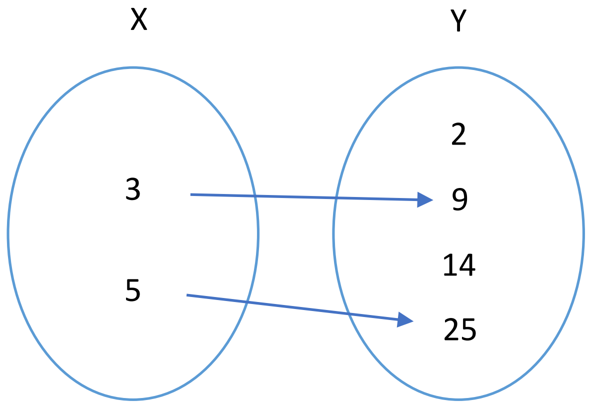


The arrow in the diagram represents the relation 'took the subject'.

Domain (Input): Students
Codomain (Output): Subjects

Clara took GP, Physics and Math
Sarah took GP and History

Functions



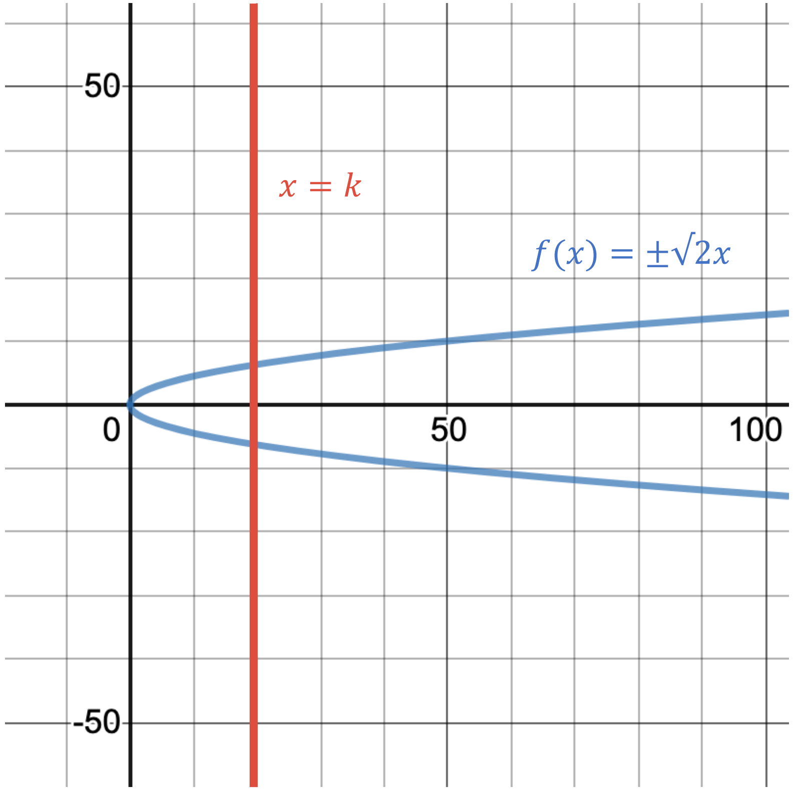
f : x |→ x^2, x = {3,5}

Domain (Input) = {3,5}
Codomain (Output) = {2,9,14,25}
Range = {9,25}

*Range is a subset of codomain

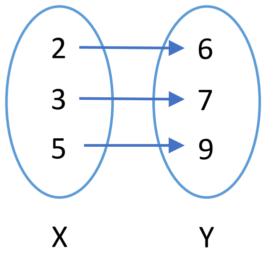
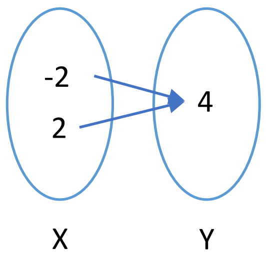
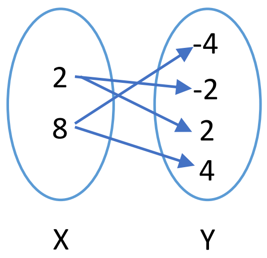
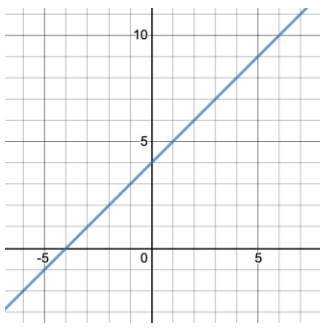
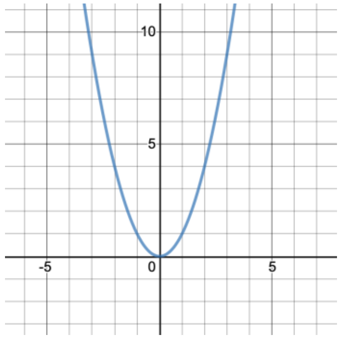
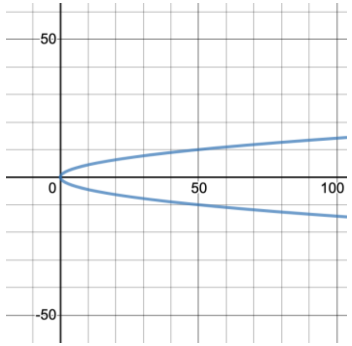
Vertical Line Test for Existence of Functions

A relation is a function if and only if any vertical line *x* = *k*, where *k* is a constant, $k \in D_f$ cuts the graph at **one and only one** point.



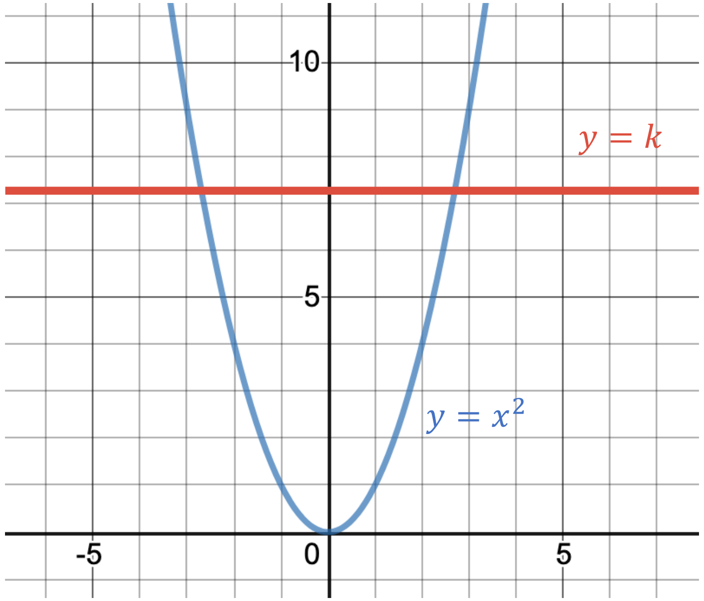
Since the vertical line *x* = *k* , where $k \in \mathbb{R}^{+}_{0}$ does not cut the graph of f at one and only one point, f is not a function.

Functions

	Functions		Relations
	One-to-One	Many-to-One	One-to-Many
Definition	No two elements in the given domain have the same image (output).	Two or more elements in the given domain has the same image (output).	One element in the given domain has two or more images (output).
Arrow Diagram	$f(x) = x + 4$ 	$f(x) = x^2$ 	$f(x) = \pm\sqrt{2x}$ 
Graph			

Horizontal Line Test for Existence of Inverse Functions

At A Levels, to determine if the inverse function exists, we just need to check if it is one-to-one. To do this, we use the horizontal line test. Graphically, f is one-one if and only if every horizontal line $y = k$, $k \in \mathbb{R}$ cuts the graph of f at **one and only one** point.



Since the horizontal line $y = k$ cuts the graph at more than one point, it is not a one-to-one function. Therefore, its inverse does not exist.

Note: We can also test if a function is inverse or one-to-one algebraically.
If $f(x_1) = f(x_2) \rightarrow x_1 = x_2$ for $x_1, x_2 \in D_f$, then f is one-one.

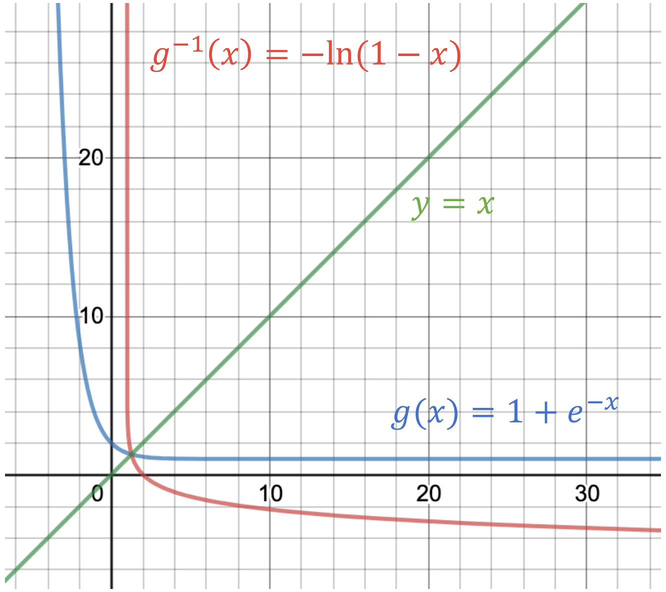
Inverse Functions

The ***inverse function*** of f, f^{-1}, is such that if $y = f(x)$, then $x = f^{-1}(y)$.

Domain of f^{-1} = Range of f
Range of f^{-1} = Domain of f

Relationship between a Functions and its Inverse

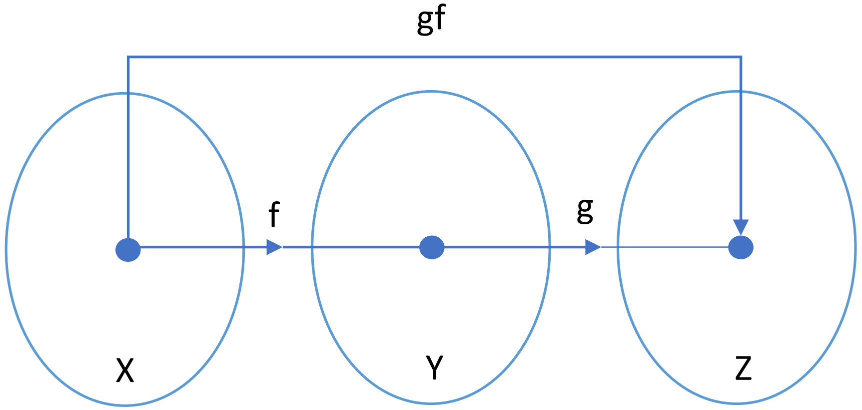
Graphically, a function and its inverse are reflected images of each other in the line $y = x$.
The point(s) of intersection of g and g^{-1} lie(s) on the line $y = x$.
When sketching the graphs of g and g^{-1} on the same axis, the scales of the x-axis and y-axis must be the same.



Composite Functions

The **composite function** gf is defined as g f (x) = g [f (x)]
If f is a function which maps set X to set Y and g is a function which maps set Y to set Z, then the composite function gf maps set X to set Z

gf denotes applying the mapping of the inner function f followed by the mapping of the outer function g.



Domain of Composite Functions: $D_{gf} = D_f$

The range of gf is the range of g whose domain has been restricted to the range of f. **To find the range of gf:**

1. Use the domain of f (inner function) to find the range of f
2. Set the range of f to be the new domain of g (outer function)
3. With this new domain of g, find the new range of g
4. This new range of g is the range of gf

Existence of a Composite Function

For the composite function gf to exist, the range of the inner function f must be a subset of the domain of the outer function g

Range of f ⊆ Domain of g

i.e. $R_f \subseteq D_g$

Composition of a Functions and its Inverse

For inverse functions f and f^{-1}:

$ff^{-1}(x) = f^{-1}f(x) = x$

The domain of $f^{-1}f(x)$ is D_f

The domain of $ff^{-1}(x)$ is $D_{f^{-1}}$